Coverage Statistics of Distributed Sensor Fields with Heterogeneous Range Sensitivity

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PREFACE

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This report presents analytical results for the coverage provided by a system of randomly distributed sensors having heterogeneous range sensitivities. The need to rapidly deploy even a moderately large number of sensors on short notice could place limitations on sensor quality. In particular, the requirement that all sensors have the same range sensitivity could pose a serious problem to the timely deployment of the sensor network. Through an entirely probabilistic approach, coverage statistics for a system of randomly distributed sensors having heterogeneous range sensitivities are obtained. Simulation studies that support the theoretical results are provided. Throughout this report, there are remarks on the implications of the analytical results on design guidance and sensor deployment.

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1. INTRODUCTION

This report examines the detection performance of a distributed sensor field composed of a moderate to large number of sensors having heterogeneous range sensitivities. The need to rapidly deploy a large number of sensors on short notice could place limitations on sensor quality. In particular, the requirement that all sensors have the same range sensitivity could be economically prohibitive for even moderately large systems. This report examines the coverage statistics for a system of randomly distributed sensors having heterogeneous range sensitivities. Throughout, comments are made on the implications of the analytical results on design guidance and sensor deployment.

This research differs from previous work in this area in that it allows for the range sensitivity of the sensors to be independent; it requires only the distributional characteristics of the range sensitivities to be known.* Other work in this area considers the coverage provided by systems of sparsely distributed sensors, all of which have identical range sensitivity; see for example, Wettergren [1] or Cox [2]. In a recent article, Wan and Yi [3] examine the coverage provided by randomly distributed wireless sensor networks. However, their research considers only homogeneous sensor systems, that is, systems in which all sensors have identical range sensitivity. This analysis differs also from previous research in that it considers the problem in both two- and three-dimensional space; previous work in the area considered only the two-dimensional problem.

Section 2 of this report begins with a brief review of temporal Poisson point processes, followed by a discussion of spatial Poisson point processes. The main result on the coverage of a randomly distributed sensor field is given in section 3. Section 4 contains examples demonstrating how the theory can be used to measure sensor field coverage. Section 5 compares the coverage of a sensor field with heterogeneous range sensitivities to the coverage of a sensor field in which all sensors have identical sensitivity range. Section 6 is a simulation study that verifies the theoretical results. Finally, appendix A contains a conditional uniformity of sensor locations and appendix B contains a proof of the main result.

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*It is supposed that the distributional characteristics of the sensors will be provided by the sensor manufacturer or can be determined through sampling.

2. SPATIAL POISSON PROCESSES

This section commences with a review of the more familiar temporal Poisson point process, followed by a description of its spatial equivalent. Recall that the usual (i.e., temporal) Poisson point process N_t, with rate λ, is a stochastic process in which N_t is the number of occurrences of an event during t units of time. Moreover, N_t follows a Poisson distribution with mean λt. Finally, the event occurrences in disjoint time intervals are independent. More formally, the counting process $\mathbf{N} = \{N_t, t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, if it possesses the following properties:[*]

1. $N_0 = 0$,

2. $P\{N_h = 1\} = \lambda h + o(h)$,

3. $P\{N_h \geq 2\} = o(h)$,

4. N_t satisfies the stationary and independent increment property.

Property 2 says that, over a very short time period, the probability of an event occurrence is approximately linear with time. Property 3 says that the probability of two or more occurrences over a very short time period is essentially zero. And according to property 4, the process N_t is independent of the process $N_{t+h} - N_t$, i.e., the number of occurrences up to time t is independent of the number of occurrences after time t.

For a spatial Poisson process, instead of counting the number of occurrences over a time interval, one is interested in the number of points associated with an event over a region of space. Let $\mathcal{S}$ be an n-dimensional set and suppose $A \subset \mathcal{S}$, i.e., A is a subset of $\mathcal{S}$. Consider points scattered randomly throughout $\mathcal{S}$ and let $N(A)$ denote the number of points from the scattered set that are contained in A. The stochastic process $N(A)$ is called a *point process* in $\mathcal{S}$. Depending on the dimension of $\mathcal{S}$, let $\|A\|$ denote the length, area, volume, etc., of A. The stochastic process $\mathbf{N} = \{N(A), A \subset \mathcal{S}\}$ is an n-dimensional Poisson counting process with parameter $\lambda > 0$ if

(a) $N(A)$ follows a Poisson distribution with mean $\lambda\|A\|$, and

(b) the number of points occurring in disjoint subsets of $\mathcal{S}$ are mutually independent.

Thus, for the spatial Poisson process, the statements equivalent to properties $1 - 4$ above are (here $\emptyset$ denotes the empty set):

---

[*]The function $g(h)$ is said to be *little-oh* h, written $g(h) = o(h)$ if $\lim_{h \to 0} \frac{g(h)}{h} = 0$. Thus, according to property 3, as h goes to zero, the probability of two or more occurrences goes to zero faster than any linear function.

1. $N(\emptyset) = 0$,

2. $P\{N(A) = 1\} = \lambda\|A\| + o(\|A\|)$,

3. $P\{N(A) \geq 2\} = o(\|A\|)$,

4. $N(A)$ satisfies the stationary and independent increment property.

In particular, property (2) says that if $\|A\|$ (the volume of the set A) is small, then the probability of one occurrence is approximately linear with respect to volume; property (3) says that for small volumes, the probability of two or more occurrences is approximately zero. Finally, property (4) says that if $A, C \subset \mathcal{S}$, then $N(A)$ is independent of $N(C \cap A^c)$.

The definitions are summarized as follows: If $\mathbf{N} = \{N(A), A \subset \mathcal{S}\}$ is an n-dimensional Poisson counting process with parameter $\lambda > 0$, then the probability of k occurrences in the subset A is given by*

$$P\{N(A) = k\} = \frac{(\lambda\|A\|)^k e^{-\lambda\|A\|}}{k!}.$$

See, for example, Resnick [4] or Kingman [5] for further information on spatial Poisson processes.

---

*Recall that if X is a Poisson random variable with mean λ, then $P\{X = k\} = \lambda^k e^{-\lambda}/k!$. According to property (a), therefore, one can substitute $N(A)$ for X and $\lambda \| A \|$ for λ to get the equivalent probabilities for the spatial random variable.

3. SENSOR COVERAGE

This section contains the main result of this report, with the remainder of the report addressing the consequences of this result. Most of the discussion concerns coverage in three-dimensional space; however, all arguments apply to two dimensions also. A two-dimensional equivalent of the main result is also stated. The proof is technical and is postponed until appendix B. A restatement of the problem can be found in chapter 16 of Karlin and Taylor [6], where instead of sensors and range sensitivities, spherical centers and corresponding radii are considered.

Theorem 1: Consider a set of omnidirectional sensors in three-dimensional space scattered throughout some region $S \subset R^3$. Suppose that the sensors are spatially distributed according to a Poisson point process with parameter λ. Suppose also that the range sensitivity of all sensors is distributed according to the cumulative distribution function $F(r)$ with density $F'(r) = f(r)$ having finite third moment. Finally, assume that range sensitivity is independent of the sensor location. Then, the number of sensors that will detect a target located at some point $\mathbf{x} \in S$ is a Poisson random variable with mean

$$\Lambda = \frac{4}{3}\lambda\pi \int_0^\infty r^3 f(r)dr. \tag{1}$$

Remark 1: Given an arbitrary point $\mathbf{x} \in S \subset R^n$,

$$P\{\text{target at position } \mathbf{x} \text{ detected by } k \text{ sensors}\} = \frac{\Lambda^k e^{-\Lambda}}{k!}.$$

The probability that the point is not detected is $\exp(-\Lambda)$. Hence, the probability that at least one sensor covers the point is $\rho = 1 - \exp(-\Lambda)$.

Definition 1: Throughout the remainder of this report, the statistic $\rho = 1 - \exp(-\Lambda)$ will be referred to as the *predicted coverage* for the sensor network. The coverage statistic ρ is a measure of the probability that an arbitrary point is detected by at least one sensor.

Remark 2: If an arbitrary point in the volume is to be detectable by at least some fixed number of sensors, then the result shows us what intensity level (i.e., Λ) is required to achieve this. Since the mean number of sensors that detect a target at some point in S is $\frac{4}{3}\lambda\pi \int_0^\infty r^3 f(r)dr$, one can adjust λ (by increasing the number of sensors scattered throughout S) to achieve the

desired coverage. The only quantity that is required is some estimate of $f(r)$, the range sensitivity density.

Remark 3 [The case of fixed range sensitivity]: The reference to range sensitivity distribution alludes to the fact that the sensors are not perfect. Thus, instead of a few very expensive sensors with fixed range sensitivity r_o, the network may contain several less expensive sensors with variable range sensitivities. Nevertheless, through sampling, one can get an estimate of the sensitivity distribution. In the idealized case in which all sensors have identical fixed range sensitivity r_o, one obtains

$$\Lambda = \frac{4}{3}\lambda\pi \int_0^\infty r^3 \delta(r - r_o)dr$$
$$= \lambda\frac{4}{3}\pi r_o^3.$$

3.1 NEAREST-NEIGHBOR PROBLEMS

Suppose that sensors are distributed throughout some three-dimensional space with unknown or slowly changing* Poisson parameter λ. If the sensors are designed to *communicate* with each other so that it is possible to estimate the average distance between them at any time, then, as will be shown presently, this result can be used to determine the current value of λ.

Proposition: Consider omnidirectional sensors in R^3 distributed according to a Poisson process with intensity parameter λ. The distribution function $F_D(r)$ of the distance between a sensor and its nearest neighbor, and the mean distance $E(D)$ between nearest neighbors are, respectively,[†]

$$F_D(r) = 1 - \exp\left\{-\lambda\frac{4}{3}\pi r^3\right\} \quad \text{and} \quad E(D) = \frac{\Gamma(\frac{1}{3})}{\sqrt[3]{36\lambda\pi}}.$$

Proof: Let A be the region of space within a distance r of the reference sensor. Then,

$$
\begin{aligned}
F_D(r) &= P\left\{\text{nearest neighbor within distance } r\right\} \\
&= 1 - P\left\{\text{nearest neighbor not within distance } r\right\} \\
&= 1 - P\left\{\text{no other sensor within distance } r \text{ of this one}\right\} \\
&= 1 - \exp\left\{-\lambda\|A\|\right\}\frac{(\lambda\|A\|)^0}{0!} \\
&= 1 - \exp\left\{-\lambda\frac{4}{3}\pi r^3\right\}.
\end{aligned}
$$

---

*For example, the parameter λ may change due to ocean currents.
[†]Recall that $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$.

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The nearest-neighbor density $f_D(r) = F'_D(r) = 4\lambda\pi r^2 \exp\left\{-\lambda\frac{4}{3}\pi r^3\right\}$. Therefore,

$$
\begin{aligned}
E(D) &= \int_0^\infty 4\lambda\pi r^3 \exp\left\{-\lambda\frac{4}{3}\pi r^3\right\} dr \\
&= \frac{3}{\sqrt[3]{36\lambda\pi}} \int_0^\infty u^{1/3} e^{-u} du \\
&= \frac{3\Gamma(\frac{4}{3})}{\sqrt[3]{36\lambda\pi}} \\
&= \frac{\Gamma(\frac{1}{3})}{\sqrt[3]{36\lambda\pi}},
\end{aligned}
\tag{2}
$$

where the identity $\Gamma(r+1) = r\Gamma(r)$ for $r > 0$ is used in the last line.

Knowing λ enables one to determine the probability of detecting an object that enters the sensor network space: From formula (1), the number of sensors that detect the object will be a Poisson random variable with mean

$$
\begin{aligned}
\Lambda &= \frac{4}{3}\lambda\pi \int_0^\infty r^3 f(r) dr \\
&= \frac{1}{27} \frac{\Gamma^3(\frac{1}{3})}{[E(D)]^3} \int_0^\infty r^3 f(r) dr,
\end{aligned}
\tag{3}
$$

where the substitution $\lambda = \frac{1}{36\pi} \frac{\Gamma^3(\frac{1}{3})}{[E(D)]^3}$ obtained from equation (2) was made.

Before concluding this section, it should be pointed out that the proposition's results relating an arbitrary sensor to its nearest-neighbor sensor also apply to any arbitrary point. That is, the distribution function $G_D(r)$ of the distance between an arbitrary point and the nearest sensor, and the mean distance $E(\mathcal{D})$ between an arbitrary point and the nearest sensor are, respectively,

$$
G_D(r) = 1 - \exp\left\{-\lambda\frac{4}{3}\pi r^3\right\} \quad \text{and} \quad E(\mathcal{D}) = \frac{\Gamma(\frac{1}{3})}{\sqrt[3]{36\lambda\pi}}.
$$

3.2 PROBLEM STATEMENT FOR R^2

Theorem 2: Consider discs in two-dimensional space with centers distributed according to a Poisson distribution with mean $\lambda\|A\|$, where $\|A\|$ represents the area of the set A. Suppose

that the radii of all discs are independent of the location of the center of the disc and distributed according to $F(r)$ with density $f(r)$ and finite second moment. Then, the number of discs that cover a point $\mathbf{x}$ is a Poisson random variable with parameter

$$\lambda\pi \int_0^\infty r^2 f(r)dr. \tag{4}$$

The proof of this result follows the proof of Theorem 1 provided in appendix B.

Proposition: For the two-dimensional case, the solutions to the nearest-neighbor problem are

$$F_D(x) = 1 - \exp\left\{-\lambda\pi x^2\right\} \quad \text{and} \quad E(D) = 1/(2\sqrt{\lambda}),$$

respectively.

4. EXAMPLES

Example 1: Suppose sensors have been distributed throughout some region in space with average distance between a sensor and its nearest neighbor equal to E(D) kilometers. (a) What is the probability that a target in the region is detected by at least N sensors? (b) What is the probability that a target in the region is detected by at least two sensors?

Solution (a):

$$P\{X \geq N\} = \sum_{n=N}^{\infty} \frac{\Lambda^n \exp\{-\Lambda n\}}{n!}$$

$$= 1 - \sum_{n=0}^{N-1} \frac{\Lambda^n \exp\{-\Lambda n\}}{n!}, \tag{5}$$

where $\Lambda = \frac{1}{27} \frac{\Gamma^3(\frac{1}{3})}{[E(D)]^3} \int_0^{\infty} r^3 f(r) dr$.

Solution (b): Let X denote the number of sensors that detect the target:

$$
\begin{aligned}
P\{X \geq 2\} &= 1 - (P\{X = 0\} + P\{X = 1\}) \\
&= 1 - e^{-\Lambda} - \Lambda e^{-\Lambda} \\
&= 1 - (1 + \Lambda)e^{-\Lambda} \\
&= 1 - \frac{\left(1 + \frac{1}{27}\frac{\Gamma^3(\frac{1}{3})}{[E(D)]^3}\int_0^{\infty} r^3 f(r) dr\right)}{\exp\left(\frac{1}{27}\frac{\Gamma^3(\frac{1}{3})}{[E(D)]^3}\int_0^{\infty} r^3 f(r) dr\right)}.
\end{aligned}
$$

Example 2: (a) What average distance is required between each sensor and its nearest neighbor in order to have an α-% chance that at least two sensors cover an arbitrary point in the volume? (b) How intensely* should the sensors be distributed over the volume in order to achieve this coverage?

Solution (a): Here, we want

$$
\begin{aligned}
P\{X \geq 2\} &= 1 - (P\{X = 0\} + P\{X = 1\}) \\
&= 1 - e^{-\Lambda} - \Lambda e^{-\Lambda} \\
&= \alpha.
\end{aligned}
$$

---

*This refers to the parameter λ: larger values of λ imply a denser sensor field.

Thus, one first solves $\Lambda - \log(1 + \Lambda) + \log(1 - \alpha) = 0$ for Λ (numerically) and then uses equation (3) to obtain

$$E(D) = \frac{\Gamma(\frac{1}{3})}{3\Lambda^{1/3}} \sqrt[3]{\int_0^\infty r^3 f(r) dr}. \tag{6}$$

Having found the answer to (a), one can easily answer (b): Solving (2) for λ gives

$$\lambda = \frac{1}{36\pi} \left[\frac{\Gamma(\frac{1}{3})}{E(D)} \right].$$

5. COVERAGE COMPARISON STUDY

This section provides a comparison of the coverage provided by two systems of randomly distributed sensors. Throughout this section, it is assumed that the sensors are distributed over a planar region, i.e., $S \subset R^2$. Case 1 considers a system in which all sensors have identical range sensitivity: $r_o = 3$ units. Thus, the range sensitivity density for this system is the dirac-delta function $\delta(r - r_o)$. For case 2, the gamma density is used to model the range sensitivity over all sensors, that is,

$$f(r) = \frac{1}{\Gamma(\alpha)\beta^\alpha} r^{\alpha-1} e^{-r/\beta}, \qquad r \geq 0.$$

In the example given, $\alpha = 2.5$ and $\beta = 1.2$; thus, the mean range sensitivity is also $\alpha\beta = 3.0$. The gamma density is used to model range sensitivity to convey the idea that some sensors will be better than others. Upon deployment, some sensors will be defective or give very poor performance, whereas others will perform very well. And, of course, several will provide average performance (figure 1).
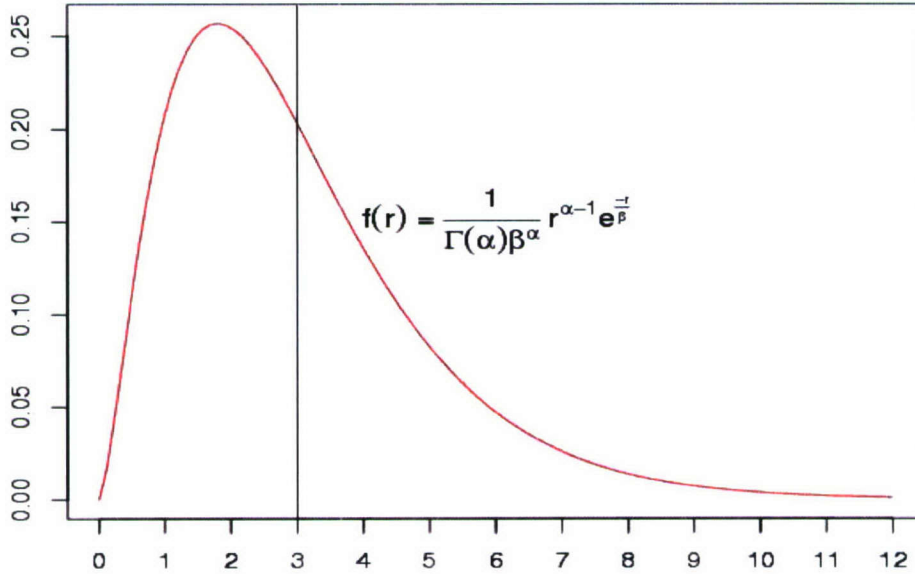

$$f(r) = \frac{1}{\Gamma(\alpha)\beta^\alpha} r^{\alpha-1} e^{\frac{-r}{\beta}}$$

Figure 1. Gamma Density Function with Parameters $\alpha = 2.5$ and $\beta = 1.2$
(The line $x = 3$ represents the idealized density with support concentrated at 3.)

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In general, for f equal to the gamma density, one has[*]

$$
\begin{aligned}
\Lambda &= \frac{C}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty r^{\alpha+\gamma-1} e^{-r/\beta} dr \\
&= \frac{C}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha+\gamma)\beta^{\alpha+\gamma}. \tag{7}
\end{aligned}
$$

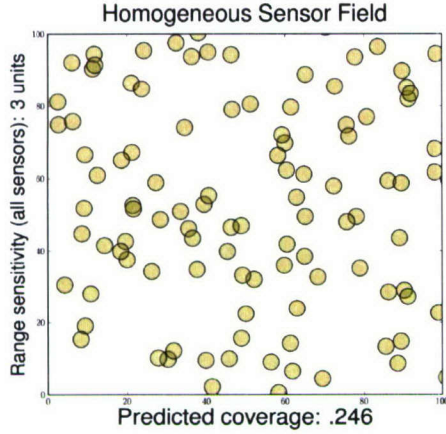From section 3.2, it follows that the mean parameter for the homogeneous case is $\Lambda = \lambda\pi \int_0^\infty r^2 \delta(r - r_o)dr = \lambda\pi r_o^2$.

In this simulation, 100 sensors are uniformly distributed over a 100 by 100 square unit area. For the homogeneous system, with all sensors having range up to $r_0 = 3$ units, the predicted coverage[†] is $\rho = 1 - \exp(-\lambda\pi r_0^2) = 0.246$. On the other hand, for the heterogeneous system with range sensitivity following a gamma distribution with parameters $\alpha = 2.5, \beta = 1.2$, the predicted coverage is $\rho = 1.0 - \exp\left\{-\frac{\lambda\pi}{\Gamma(\alpha)\beta^\alpha}\Gamma(\alpha+2)\beta^{\alpha+2}\right\} = 0.326$. Note that the mean sensitivity range for the heterogeneous sensors is $\alpha\beta = 3$. Figure 2 compares the coverage provided by just one simulation of a randomly distributed system of homogeneous and heterogeneous sensors. The plot on the left depicts 100 randomly placed sensors. The actual coverage provided by this system is 0.256. The plot on the right contains 100 sensors with 0.339 coverage. This is, of course, just one instantiation of the two systems. In the next section, data from several instantiations are used to test the long-term accuracy of the simulations.

---

[*]Here, the parameters C and γ depend on the dimension: if the search region is in R^2, then $\gamma = 2$ and $C = \lambda\pi$; if the search region is . a subset of R^3, then $\gamma = 3$ and $C = \frac{4}{3}\lambda\pi$.
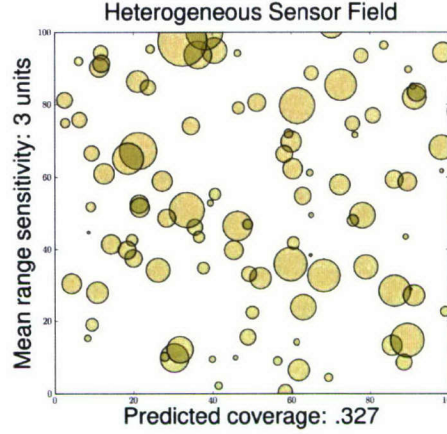
[†]The term *predicted coverage* denotes the probability that an arbitrary point in the space is within range of at lease one sensor.

100 Homogeneous Sensors

100 Heterogeneous Sensors

Figure 2. Comparison of Homogeneous and Heterogeneous Sensor Fields

Figure 3 compares the number of sensors deployed to the predicted coverage. From the formula for ρ, it is clear that the critical factor in the coverage statistic (for the case of a planar field of sensors*) is the mean coverage area of the sensors: $\int_0^\infty r^2 f(r)dr$.
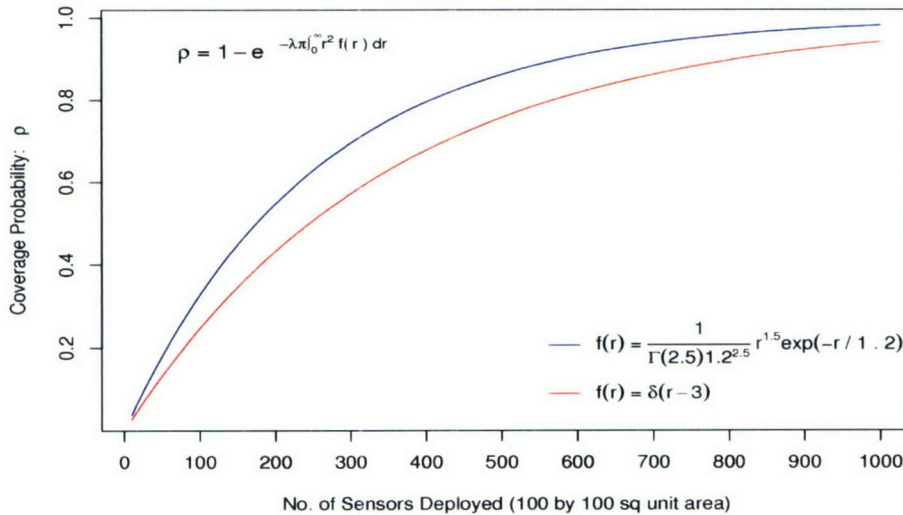

Figure 3. Number of Sensors Deployed Versus the Probability of Detection
by at Least One Sensor

---

*For a network of sensors over a volume, the mean volume $\int_0^\infty r^3 f(r)dr$ is the critical statistic.

6. MONTE CARLO STUDY OF THE COVERAGE STATISTIC

Suppose that, for each simulation, one randomly distributes sensors over a search region and then randomly chooses a point in this region. The randomly chosen point should be covered by at least one sensor approximately $\rho = 1 - \exp(-\Lambda)$ percent of the time.[*] For the ith simulation, let $X_i = 1$ if the randomly chosen point is within range of at least one sensor; otherwise, set $X_i = 0$. The random variables X_i are binomial with success probability ρ. The sample average $\hat{\rho} = \frac{1}{N}\sum_{i=1}^{N} X_i$ of the simulations can be used to determine the number of simulations required so that $\hat{\rho}$ is within 5% of ρ with 90% confidence. Specifically, one wants to estimate N such that

$$P(\mid \hat{\rho} - \rho \mid \leq 0.05\rho) \;\; = \;\; 0.90\,.$$

Note that

$$\mu_{\hat{\rho}} \;\; = \;\; \rho$$
$$\sigma_{\hat{\rho}}^2 \;\; = \;\; \rho(1-\rho)/N.$$

By the Central Limit Theorem, for N large, $\hat{\rho}$ is approximately normally distributed with mean ρ and variance $\rho(1-\rho)/N$. Hence,

$$
\begin{aligned}
0.90 \;\; &= \;\; P(\mid \hat{\rho} - \rho \mid \leq 0.05\rho) \\
&= \;\; P\left(\left| \frac{\hat{\rho} - \rho}{\sqrt{\rho(1-\rho)/N}} \right| \leq 0.05\rho \frac{\sqrt{N}}{\sqrt{\rho(1-\rho)}} \right) \\
&= \;\; P\left(\mid Z \mid \leq 0.05\rho \frac{\sqrt{N}}{\sqrt{\rho(1-\rho)}} \right),
\end{aligned}
$$

where Z is the standard normal random variable. In order for $\hat{\rho}$ to be within 5% of ρ (90% of the time), it is required that

$$0.05\rho \frac{\sqrt{N}}{\sqrt{\rho(1-\rho)}} \;\; \approx \;\; 1.645;$$

that is, $N \approx (1.645/0.05)^2(1-\rho)/\rho$.

---

[*]Recall from Theorem 1 that the number of sensors that detect a randomly chosen point in space is Poisson with mean Λ. Hence, the probability that the point is not detected is $\exp(-\Lambda)$. The probability that at least one sensor covers the point is $1 - \exp(-\Lambda)$.

The analysis was performed for $\lambda = 0.05$ over a 100 by 100 square unit area. Thus, each simulation has roughly $10000\lambda = 500$ sensors. As in the previous section, each sensor in the homogeneous field has a range sensitivity of $r_0 = 3$ units, and the sensor range density for the heterogeneous system is gamma with parameters $\alpha = 2.5, \beta = 1.2$. The predicted coverages provided by the two systems are 0.757 and 0.862, respectively. Note that, for this example, detection in the homogeneous case is a more rare event, hence more simulations are required to approximate the true value: $N = (1.645/0.05)^2(1 - 0.757)/0.757 = 348$ versus $N = (1.645/0.05)^2(1 - 0.862)/0.862 = 174$ for the heterogeneous case.

Table 1. Monte Carlo Simulation of Coverage Probability

System	No. Simulations	ρ	$\hat{\rho}$	Percentage Difference
Homogeneous	348	0.757	0.744	1.72
Heterogeneous	174	0.862	0.845	1.97

7. SUMMARY

Through an entirely probabilistic approach, coverage statistics for a system of randomly distributed sensors having heterogeneous range sensitivities have been derived in this report. It has been demonstrated that the case in which all sensors have identical range sensitivity is just one idealized case of the heterogeneous problem. Simulation studies that support the theoretical results are provided.

APPENDIX A

CONDITIONAL UNIFORMITY AND INDEPENDENCE

Lemma: Let N be a spatial Poisson process with parameter λ. Then, under condition $N(A) = k$ for $\|A\| > 0$, the k points are independent and uniformly distributed.

Proof: Let $A_1, A_2, \ldots, A_n$ be n disjoint regions with $\bigcup_1^n A_i = A$. Suppose also that the integers $k_1, k_2, \ldots k_n$ satisfy $k_1 + k_2 + \ldots + k_n = k$. Then,

$$
\begin{aligned}
P\{N(A_1) &= k_1, N(A_2) = k_2, \ldots, N(A_n) = k_n \mid N(A) = k\} \\
&= \frac{P\{N(A_1) = k_1, N(A_2) = k_2, \ldots, N(A_n) = k_n\}}{P\{N(A) = k\}} \\
&= \frac{\left[\frac{[\lambda\|A_1\|]^{k_1}}{k_1!}e^{-\lambda\|A_1\|}\right]\left[\frac{[\lambda\|A_2\|]^{k_2}}{k_2!}e^{-\lambda\|A_2\|}\right]\cdots\left[\frac{[\lambda\|A_n\|]^{k_n}}{k_n!}e^{-\lambda\|A_n\|}\right]}{\frac{[\lambda\|A\|]^k}{k!}e^{-\lambda\|A\|}} \\
&= \frac{k!}{k_1!k_2!\ldots k_n!}\left[\frac{\|A_1\|}{\|A\|}\right]^{k_1}\left[\frac{\|A_2\|}{\|A\|}\right]^{k_2}\cdots\left[\frac{\|A_n\|}{\|A\|}\right]^{k_n}.
\end{aligned}
\tag{A-1}
$$

In particular,

$$
\begin{aligned}
P\{N(A_1) = 1 \mid N(A) = 1\} &= P\{N(A_1) = 1, N(A_1^c) = 0 \mid N(A) = 1\} \\
&= \frac{P\{N(A_1) = 1, N(A_1^c) = 0\}}{P\{N(A) = 1\}} \\
&= \frac{1!}{1!0!}\left[\frac{\|A_1\|}{\|A\|}\right]^1\left[\frac{\|A_1^c\|}{\|A\|}\right]^0 \\
&= \frac{\|A_1\|}{\|A\|}.
\end{aligned}
\tag{A-2}
$$

According to (A-2), *given that an event has occurred in volume A, it is equally likely to be found anywhere in A.* Also, note that

$$
P\{N(A_1) = k_1 \mid N(A) = k_1\} = \left[\frac{\|A_1\|}{\|A\|}\right]^{k_1}.
\tag{A-3}
$$

Thus, equation (A-1) implies independence. Note that the leading coefficient in (A-1) is the sum over all the configurations in which the k points can be divided into n subgroups with k_1 points in the first group, k_2 in the second, etc. Each of these configurations has probability $\|A_1\|^{k_1}\|A_2\|^{k_2}\ldots\|A_n\|^{k_n}/\|A\|^k$.

APPENDIX B
PROOF OF MAIN THEOREM

Theorem 1: Consider a set of omnidirectional sensors in three-dimensional space scattered throughout some region $\mathcal{S} \subset R^3$. Suppose that the sensors are spatially distributed according to a Poisson point process with parameter λ. Suppose also that the range sensitivity of all sensors is distributed according to the cumulative distribution function $F(r)$ with density $F'(r) = f(r)$ having finite third moment. Finally, assume that range sensitivity is independent of the sensor location. Then, the number of sensors that will detect a target located at some point $\mathbf{x} \in \mathcal{S}$ is a Poisson random variable with mean

$$\Lambda = \frac{4}{3}\lambda\pi \int_0^\infty r^3 f(r)dr.$$

Independence Assumptions: (1) Sensors are distributed according to a Poisson random variable. *This means that the number of sensors in non-intersecting regions of space (e.g., non-intersecting spherical shells centered about any point) are independent.* It is useful to think of each sensor and its sensitivity extent as a sphere, with the center of the sphere at the sensor location, and radius equal to the sensitivity range of the sensor. (2) According to the theorem, the radius of each sphere is assumed to be a random variable independent of the location of the sphere.

Proof: Fix the origin* at any point in R^3. Let $S(r)$ be the sphere of radius r with center at the origin. Also, let $S(r, r + \Delta r)$ denote the volume of the shell, or region, between two concentric spheres centered at the origin and having radii r and $r + \Delta r$, respectively. Then,

$$
\begin{aligned}
S(r, r + \Delta r) &= \|S(r + \Delta r)\| - \|S(r)\| \\
&= \frac{4}{3}\pi(r + \Delta r)^3 - \frac{4}{3}\pi r^3 \\
&= \frac{4}{3}\pi \left[r^3 + 3r^2\Delta r + 3r(\Delta r)^2 + (\Delta r)^3 - r^3\right] \\
&= \frac{4}{3}\pi\Delta r \left[3r^2 + 3r\Delta r + (\Delta r)^2\right] \\
&= 4\pi r^2 \Delta r + o(\Delta r).
\end{aligned}
$$

Therefore, the probability of a sphere occurring in the shell (i.e., having center in the shell $S(r, r + \Delta r)$) with radius extending out to the origin is the product of (i) the probability a

---

*For argument's sake, the origin is used, although any fixed point would suffice. It might be useful for the reader to consider this point a target, as it will be shown that the number of sensors that detect this point follows a Poisson distribution.

spherical center occurs within the shell (i.e., $\lambda\left(4\pi r^2\Delta r + \o(\Delta r)\right)$) and (ii) the probability that the sphere's radius extends at least out to r (i.e., $\int_r^\infty f(\rho)d\rho$):

$$\lambda\left(4\pi r^2\Delta r + \o(\Delta r)\right)\int_r^\infty f(\rho)d\rho = \left(\lambda 4\pi r^2\int_r^\infty f(\rho)d\rho\right)\Delta r + o(\Delta r).$$

That is,

$$P\left\{N(S(r,r+\Delta r)) = 1\right\} = \Lambda(r)\Delta r + o(\Delta r), \qquad\qquad \text{(B-1)}$$

where $N(S(r,r+\Delta r))$ is the number of sensors (i.e., sphere centers) to occur in the shell $S(r,r+\Delta r)$ with radius extending out to the origin, and $\Lambda(r) = 4\pi\lambda r^2\int_r^\infty f(\rho)d\rho$.

It remains to be shown that the probability of the occurrence of two or more spheres with centers in the shell $S(r,r+\Delta r)$ and radii extending out to the origin is $o(\Delta r)$:

$$
\begin{aligned}
P\left\{N(S(r,r+\Delta r)) \geq 2\right\} &= \exp\left(-4\pi\lambda r^2\Delta r\right)\sum_{k=2}^\infty \frac{\left[4\pi\lambda r^2\Delta r\right]^k}{k!}\left(\int_r^\infty f(\rho)d\rho\right)^k \\
&= \exp\left(-4\pi\lambda r^2\Delta r\right)\sum_{k=2}^\infty \frac{\left[4\pi\lambda r^2\Delta r\int_r^\infty f(\rho)d\rho\right]^k}{k!} \\
&\leq \exp\left(-4\pi\lambda r^2\Delta r\right)\frac{\left[4\pi\lambda r^2\Delta r\int_r^\infty f(\rho)d\rho\right]^2}{2!} \\
&\quad \times \sum_{k\geq 0}\frac{\left[4\pi\lambda r^2\Delta r\int_r^\infty f(\rho)d\rho\right]^k}{k!} \\
&= \exp\left(-4\pi\lambda r^2\Delta r\right)\exp\left(-4\pi\lambda r^2\Delta r\int_r^\infty f(\rho)d\rho\right) \\
&\quad \times \frac{\left[4\pi\lambda r^2\int_r^\infty f(\rho)d\rho\right]^2}{2!}(\Delta r)^2 \qquad\qquad \text{(B-2)} \\
&= o(\Delta r). \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B-3)}
\end{aligned}
$$

Note that for small Δr, the exponential factors in (B-2) are approximately 1. The coefficient of $(\Delta r)^2$ in the third factor is a constant.

From the two independence assumptions, it follows that $S(r)$ has independent increments. Now, (B-1), (B-3) and the independent increment property imply that the number of spheres with centers approximately a distance r from the origin and having radius out to the

origin is a Poisson random variable with mean $\Lambda(r)$. That is, the number of sensors with range out to the origin is a non-homogeneous Poisson random variable with local intensity function $\Lambda(r)$. Finally, one integrates $\Lambda(r)$ over all ranges to get the cumulative intensity function:

$$
\begin{aligned}
\int_0^\infty \Lambda(r)dr &= \int_0^\infty 4\pi\lambda r^2 \int_r^\infty f(\rho)d\rho dr \\
&= \int_0^\infty f(\rho) \int_0^\rho 4\pi\lambda r^2 dr d\rho \\
&= \int_0^\infty \frac{4}{3}\pi\lambda\rho^3 f(\rho)d\rho.
\end{aligned}
$$

Since the point chosen as origin was arbitrary, the result follows for any point in R^3.

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